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SELF-SIMILAR SHAPES OF THE FREE BOUNDARY OF A NONLINEAR-VISCOUS BAND UNDER UNIAXIAL TENSION

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An equation of evolution of small perturbations of the free boundary of a nonlinear-viscous band under quasi-static uniaxial tension is derived for studying the necking problem in metals under superplasticity conditions. It is shown that the group of symmetry of this linear parabolic equation is equivalent to the group of symmetry of the linear equation of heat conduction with an arbitrary material parameter of the model. Self-similar solutions are obtained in the form of simple and complicated steady localized structures transferred together with the material of the stretched band.

Key words: *free boundary, solitary waves, nonlinear viscosity, superplasticity, group classification.*

Introduction. Large deformations in metal specimens under uniaxial tension under conditions of high-temperature creep and, especially, of superplasticity are accompanied by macroscopic second-order surface effects [1–4]. Multiple formations of stationary and movable necks with small fixed amplitudes are observed on the free surface of the deformed specimen (Fig. 1). To the best of our knowledge, no systematic studies have been performed to gain insight into this phenomenon. Figure 1 shows the dependence of D/D_0 on x/D_0 (D and D_0 are the current and initial values of the cylinder diameter, x is the longitudinal coordinate, and ε_s is the mean strain). The character of the free surface shape evolution, during which the surface perturbations are "frozen," ensures stability of the specimen tension process up to abnormally high values. Investigations of conditions responsible for initiation and sustaining of such regimes may lead to a new method of classification and identification of viscoplastic constitutive relations going beyond the limits of the specimen principle and to formulation of a superplasticity definition conceivable for mechanicians [2].

The present paper deals with the problem of an infinite nonlinear-viscous band with free side under quasi-static tension. As in [5], the nonlinear rheological relations of an incompressible Reiner–Rivlin fluid are taken as the constitutive relations. The evolution of axially symmetric small perturbations of the free boundaries is investigated by the method of small parameter, which is the ratio of the perturbation amplitude to the band width. The first term of the uniformly converging asymptotic series of the free boundary perturbation satisfies a non-autonomous parabolic linear equation containing an arbitrary parameter m, which is a parameter of material sensitivity to the strain rate. For an arbitrary parameter m, applying equivalent (leaving the structure of the Lie algebra unchanged) time-inverting transformations, one can reduce this equation to a linear heat-conduction equation that has an infinite algebra of point symmetries. Self-similar steady solutions with a localized or a distributed character are found to exist in certain ranges of the parameter m, which provide a balance between kinematic and physical nonlinearities. Stability of these solutions is not determined.

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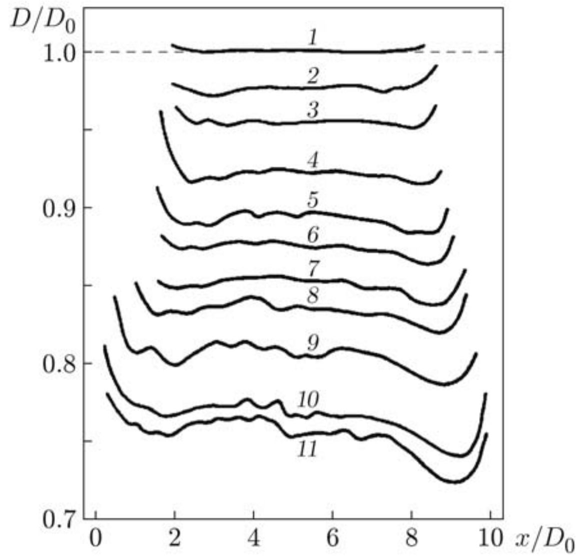

Fig. 1. Evolution of the free surface of a solid cylindrical specimen under uniaxial tension (indium–lead alloy, hot deformation, controlled parameter of the true strain rate of the order of 10^{-7} sec^{-1}) [4]: $\varepsilon_s = 0$ (1), 0.032 (2), 0.095 (3), 0.159 (4), 0.222 (5), 0.254 (6), 0.317 (7), 0.349 (8), 0.413 (9), 0.476 (10), and 0.508 (11).

Equation of Perturbations of the Free Boundary. Self-similarity of a nonlinear-viscous incompressible band under quasi-static uniaxial tension is governed by the equations

$$\sigma_{x,x} + \tau_{xy,y} = 0, \qquad \tau_{xy,x} + \sigma_{y,y} = 0; \tag{1}$$

$$\sigma_x = -p + \tau(\xi)\xi^{-1}u_{,x}, \qquad \sigma_y = -p - \tau(\xi)\xi^{-1}u_{,x}, \qquad \tau_{xy} = \tau(\xi)\xi^{-1}(u_{,y} + v_{,x})/2; \tag{2}$$

$$u_{,x} + v_{,y} = 0, \tag{3}$$

where x and y are the Cartesian orthogonal coordinates (the x axis coincides with the band axis), σ_x, σ_y, and τ_{xy} are the components of the stress tensor, p is the hydrostatic pressure, u and v are the components of displacement rate along the x and y coordinates, and $\tau = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$ and $\xi = \sqrt{(u_{,x})^2 + (u_{,y} + v_{,x})^2/4}$ are the stress and strain rate intensities energetically conjugated in the two-dimensional case; the comma means the partial derivative. The material function $\tau(\xi)$ is assumed to be an arbitrary element determining the form of the solutions for the free boundary of the stretched band. The nonlinearity of this function is responsible for the nonlinearity of system (1)–(3); the corresponding nonlinearity of the model is called the physical nonlinearity.

At the current time, the basic motion in the case of uniform tension of the band along its axis with a strain rate ξ_0 is given by

$$b_{,t} = -\xi_0 b \tag{4}$$

(b is the current transverse size of the unperturbed band). The perturbations of the rectilinear free boundaries of the band are assumed to be symmetric with respect to the band axis, which allows us to consider only one boundary $y(x,t) = b(t) + \eta(x,t)$, where $\eta < 0$ is the boundary perturbation. Using the implicit form of the equation $g = b + \eta - y \equiv 0$, we write the components of the external normal to the free boundary $n_x = g_{,x}/|\nabla g| = \eta_{,x}/|\nabla g|$ and $n_y = g_{,y}/|\nabla g| = -1/|\nabla g|$ and the static boundary conditions

$$\eta_{,x}\sigma_x - \tau_{xy} = 0, \qquad \eta_{,x}\tau_{xy} - \sigma_y = 0. \tag{5}$$

Denoting the derivative with respect to time at a fixed material particle by d/dt, we can write the condition of materiality of the free boundary $dg/dt = 0$ with allowance for Eq. (4) and relations $dx/dt \equiv u$ and $dy/dt \equiv v$:

$$\eta_{,t} + u\eta_{,x} - v - \xi_0 b = 0. \tag{6}$$

The kinematic condition (6) introduces a derivative with respect to time into the examined problem, which means that the band under quasi-static tension can be self-similar at a certain evolution of its free boundaries. We consider perturbations vanishing at infinitely distant band ends:

$$\eta(x) \to 0, \qquad x \to \pm\infty. \tag{7}$$

Substitution of Eqs. (2) and (3) into Eq. (1) leads to self-similar equations expressed in terms of the displacement rates

$$\begin{gathered}\tau^{-1}\xi p_{,x} - (m-1)[\xi^{-1}\xi_{,x}u_{,x} + \xi^{-1}\xi_{,y}(u_{,y}+v_{,x})/2] - u_{,xx} - (u_{,yy}-u_{,xx})/2 = 0,\\ \tau^{-1}\xi p_{,y} - (m-1)[-\xi^{-1}\xi_{,y}u_{,x} + \xi^{-1}\xi_{,x}(u_{,y}+v_{,x})/2] + u_{,xy} - (u_{,xy}+v_{,xx})/2 = 0,\\ \xi_{,x} = \xi^{-1}[u_{,x}u_{,xx} + (u_{,y}+v_{,x})(u_{,xy}+v_{,xx})/4], \quad \xi_{,y} = \xi^{-1}[u_{,x}u_{,xy} + (u_{,y}+v_{,x})(u_{,yy}-u_{,xy})/4],\end{gathered} \tag{8}$$

where $\tau'\xi/\tau = d\ln\tau/d\ln\xi \equiv m$ is the sensitivity to the strain rate, which is constant if $\tau(\xi)$ is a power law. For small perturbations, the coefficients at the highest derivatives in Eq. (8) are assumed to be constants corresponding to the basic motion. Therefore, we assume that $u_{,y} = v_{,x} = 0$ and $u_{,x} = \xi_0$ in the expressions for these coefficients; as a result, the system is reduced to the equations

$$2\tau^{-1}\xi p_{,x} - (2m-1)u_{,xx} - u_{,yy} = 0, \qquad 2\tau^{-1}\xi p_{,y} + (2m-1)u_{,xy} - v_{,xx} = 0. \tag{9}$$

In terms of the stream function $u = \psi_{,y}$ and $v = -\psi_{,x}$, a corollary of Eq. (9) is the equation known from the problem of stability of a viscoplastic band

$$\psi_{,xxxx} + 2(2m-1)\psi_{,xxyy} + \psi_{,yyyy} = 0,$$

which was obtained by Il'yushin [6].

Applying the method commonly used to study a layer or a band problem [7, 8], we can present the unknown fields in the form of power series in the transverse coordinate

$$p = \sum_{n=0}^{\infty} p_n(x)y^n, \qquad \psi = \sum_{n=1,3,\ldots}^{\infty} \psi_n(x)y^n,$$

$$u = \psi_{,y} = \sum_{n=1,3,\ldots}^{\infty} n\psi_n y^{n-1}, \qquad v = -\psi_{,x} = -\sum_{n=1,3,\ldots}^{\infty} \psi'_n y^n,$$

which take into account the axial symmetry of the perturbed velocity field. In the first terms of the series, we distinguish components that refer to the basic motion and perturbation:

$$p_0 = -\tau_0 + \pi, \qquad \psi_1 = \xi_0 x + \varphi$$

$[\tau_0 = \tau(\xi_0)]$. The self-similar equations (9) yield the relations between the functions p_n and ψ_n:

$$\begin{aligned} &y^0: \quad \varkappa\pi_{,x} - (2m-1)\varphi_{,xx} - 6\psi_3 = 0, \qquad p_1 = 0,\\ &y^1: \quad p_{1,x} = 0, \qquad 2\varkappa p_2 + 6(2m-1)\psi_{3,x} + \varphi_{,xxx} = 0,\\ &y^2: \quad \varkappa p_{2,x} - 3(2m-1)\psi_{3,xx} - 60\psi_5 = 0, \qquad p_3 = 0,\\ &y^3: \quad p_{3,x} = 0, \qquad 4\varkappa p_4 + 20(2m-1)\psi_{5,x} + \psi_{3,xxx} = 0,\end{aligned}$$

etc. ($\varkappa \equiv 2\tau_0^{-1}\xi_0$), which allow these functions to be resolved in terms of π and φ:

$$\begin{gathered} p = -\tau_0 + \pi + \frac{\tau_0}{4\xi_0}[(2m-1)^2 - 1]y^2\varphi_{,xxx} - \frac{2m-1}{2}y^2\pi_{,xx} + O(y^4),\\ u = \xi_0 x + \varphi - \frac{2m-1}{2}y^2\varphi_{,xx} + \frac{\xi_0}{\tau_0}y^2\pi_{,x} + O(y^4),\\ v = -\xi_0 y - \varphi_{,x}y + \frac{2m-1}{6}y^3\varphi_{,xxx} - \frac{\xi_0}{3\tau_0}y^3\pi_{,xx} + O(y^5).\end{gathered} \tag{10}$$

In what follows, the terms of the higher order of smallness than those written explicitly are omitted. Their retention leads to the emergence of additional dispersion terms in the equations of the free boundary evolution.

With allowance for Eqs. (10) and for the fact that $y = b + \eta$ on the free surface, the boundary conditions (5) and (6) in the dimensionless variables

$$x = b\bar{x}, \quad \eta = b\bar{\eta}, \quad y = b(1+\bar{\eta}), \quad t = \xi_0^{-1}\bar{t}, \quad \varphi = \xi_0 b\bar{\varphi}, \quad \pi = \tau_0\bar{\pi}$$

take the form of a system of three equations with respect to three unknown functions $\eta(x,t)$, $\varphi(x,t)$, and $\pi(x,t)$ (the bar is omitted):

$$\begin{gathered} \eta_{,x}(2-\pi+\varphi_{,x}) + m(1+\eta)\varphi_{,xx} - (1+\eta)\pi_{,x} = 0, \\ \eta_{,x}(-m(1+\eta)\varphi_{,xx} + (1+\eta)\pi_{,x}) + \varphi_{,x} + \pi = 0, \\ \eta_{,t} + \eta_{,x}(x + \varphi - (2m-1)(1+\eta)^2\varphi_{,xx}/2 + (1+\eta)^2\pi_{,x}) + \eta + (1+\eta)\varphi_{,x} \\ -(2m-1)(1+\eta)^3\varphi_{,xxx}/6 + (1+\eta)^3/3\pi_{,xx} = 0. \end{gathered} \tag{11}$$

At small perturbations, system (11) is reduced to a weakly nonlinear form by the multiscale method [9]. For this purpose, the variables are represented in the form of expansions in terms of the small parameter ε:

$$\eta = \varepsilon^s\eta_0 + \varepsilon^{2s}\eta_1 + \ldots, \qquad \varphi = \varepsilon^q\varphi_0 + \varepsilon^{2q}\varphi_1 + \ldots, \qquad \pi = \varepsilon^p\pi_0 + \varepsilon^{2p}\pi_1 + \ldots \tag{12}$$

(η_0, φ_0, π_0, η_1, φ_1, and π_1 are functions of independent variables x, t, $\chi \equiv \varepsilon x$, and $\tau \equiv \varepsilon t$, which have the order of ε^0). As a result, we can write the terms at the powers of ε [the ith column corresponds to the ith equation of system (11)]:

	$i=1$	$i=2$	$i=3$
ε^s:	$2\eta_{0,x}$	—	$\eta_{0,t} + x\eta_{0,x} + \eta_0$
ε^q:	$m\varphi_{0,xx}$	$\varphi_{0,x}$	$\varphi_{0,x} - (2m-1)\varphi_{0,xxx}/6$
ε^p:	$-\pi_{0,x}$	π_0	$\pi_{0,xx}/3$
ε^{s+1}:	$2\eta_{0,\chi}$	—	$\eta_{0,\tau} + x\eta_{0,\chi}$
ε^{q+1}:	$2m\varphi_{0,x\chi}$	$\varphi_{0,\chi}$	$\varphi_{0,\chi} - (2m-1)\varphi_{0,xx\chi}/2$
ε^{p+1}:	$-\pi_{0,\chi}$	—	$2\pi_{0,x\chi}/3$
ε^{2s}:	$2\eta_{1,x}$	—	$\eta_{1,t} + x\eta_{1,x} + \eta_1$
ε^{2q}:	$m\varphi_{1,xx}$	$\varphi_{1,x}$	$\varphi_{1,x} - (2m-1)\varphi_{1,xxx}/6$
ε^{2p}:	$-\pi_{1,x}$	π_1	$\pi_{1,xx}/3$
ε^{s+q}:	$m\eta_0\varphi_{0,xx} + \eta_{0,x}\varphi_{0,x}$	$-m\eta_{0,x}\varphi_{0,xx}$	$\eta_{0,x}\varphi_0 - (2m-1)\eta_{0,x}\varphi_{0,xx}/2 +$ $+\,\eta_0\varphi_{0,x} - (2m-1)\eta_0\varphi_{0,xxx}/2$
ε^{s+p}:	$-\eta_{0,x}\pi_0 - \eta_0\pi_{0,x}$	$\eta_{0,x}\pi_{0,x}$	$\eta_{0,x}\pi_{0,x} + \eta_0\pi_{0,xx}$
ε^{p+q}:	—	—	—

Using these terms, we can choose $s = q = p = 1$ and obtain a meaningful model:

$$\begin{gathered} 2\eta_{0,x} + m\varphi_{0,xx} - \pi_{0,x} = 0, \qquad \pi_0 + \varphi_{0,x} = 0, \\ \eta_{0,t} + x\eta_{0,x} + \eta_0 + \varphi_{0,x} - \frac{2m-1}{6}\,\varphi_{0,xxx} + \frac{1}{3}\,\pi_{0,xx} = 0; \end{gathered} \tag{13}$$

$$2\eta_{1,x} + m\varphi_{1,xx} - \pi_{1,x} = \frac{2m}{m+1}\,\eta_{0,\chi} + 2\,\frac{m+3}{m+1}\,\eta_0\eta_{0,x}, \qquad \pi_1 + \varphi_{1,x} = -\varphi_{0,\chi} - 2(\eta_{0,x})^2,$$

$$\begin{gathered} \eta_{1,t} + x\eta_{1,x} + \eta_1 + \varphi_{1,x} - \frac{2m-1}{6}\,\varphi_{1,xxx} + \frac{1}{3}\,\pi_{1,xx} = -\eta_{0,\tau} - x\eta_{0,\chi} - \varphi_{0,\chi} \\ -\frac{6m+1}{3(m+1)}\,\eta_{0,x\chi} - \eta_{0,x}\varphi_0 + \frac{2}{m+1}\,\eta_0^2 - \frac{2m+1}{m+1}\,(\eta_{0,x})^2 - \frac{2m+1}{m+1}\,\eta_0\eta_{0,xx}. \end{gathered} \tag{14}$$

Solutions in the Form of Localized and Propagating Necks. With allowance for condition (7), system (13) reduces to the following equations:

$$\eta_{0,t} + \varkappa_1\eta_0 + x\eta_{0,x} + \varkappa_2\eta_{0,xx} = 0, \qquad \varkappa_1 \equiv \frac{m-1}{m+1}, \quad \varkappa_2 \equiv \frac{2m+1}{3(m+1)}; \tag{15}$$

$$\pi_0 = -\varphi_{0,x} = \frac{2}{m+1}\,\eta_0. \tag{16}$$

Equation (15) admits the point Lie group with an infinitesimal operator $\omega\,\mathrm{e}^t\,\partial_x + \partial_t$, where ω is an arbitrary constant. The invariant $x - \omega\,\mathrm{e}^t$ of the operator leads to an exponentially self-similar substitution

$$\eta_0 = f(\zeta), \qquad \zeta \equiv x - \omega\,\mathrm{e}^t\,. \tag{17}$$

In the basic motion, an arbitrary material point having a spatial coordinate x_* in the actual (perturbed) band configuration at $t = 0$ has the spatial coordinate $x = x_*\,\mathrm{e}^t$ at any subsequent time t. Assuming in Eq. (17) that $\omega = x_*$, we can show that the contour $f(\zeta)$ is stationary in the reference system ζ moving as a rigid body together with the material point located at the origin $\zeta = 0$; in the reference system ζ, the material undergoes tension.

Substituting Eq. (17) into Eq. (15), we obtain an equation for f

$$\varkappa_2 f_{,\zeta\zeta} + \zeta f_{,\zeta} + \varkappa_1 f = 0, \tag{18}$$

which can be reduced by the transformations $f = \exp(-\bar\zeta^2/4)\bar f$ and $\zeta = \varkappa_2^{1/2}\bar\zeta$ to a standard form of the parabolic cylinder equation $\bar f_{,\bar\zeta\bar\zeta} - (\bar\zeta^2/4 + 1/2 - \varkappa_1)\bar f = 0$ [10]. The solution of the last equation can be conveniently expressed in terms of a confluent hypergeometric function Φ. As a result, the solution of Eq. (18) acquires the form

$$f(\zeta) = a(\chi,\tau)\exp\Big(-\frac{\zeta^2}{2\varkappa_2}\Big)\Phi\Big(\frac{1-\varkappa_1}{2}, \frac{1}{2}, \frac{\zeta^2}{2\varkappa_2}\Big) + b(\chi,\tau)\zeta\exp\Big(-\frac{\zeta^2}{2\varkappa_2}\Big)\Phi\Big(\frac{2-\varkappa_1}{2}, \frac{3}{2}, \frac{\zeta^2}{2\varkappa_2}\Big), \tag{19}$$

where $a \sim \varepsilon^0$ and $b \sim \varepsilon^0$. As $\zeta \to \pm\infty$, this solution has an asymptotic dependence $f(\zeta) \sim \zeta^{-\varkappa_1}$ [10]. Hence, at $m < -1$ or $m > 1$, where $\varkappa_1 > 0$ and obviously $\varkappa_2 > 0$, this solution satisfies the boundary conditions $f(\zeta) \to 0$ ($\zeta \to \pm\infty$), i.e., is localized.

For the localized functions η_0, φ_0, and π_0 to describe the perturbations η, φ, and π with accuracy to small parameters of the order ε^1, series (12) should converge uniformly, i.e., the decreasing rate of the function η_1 should be not smaller than that of η_0: $\eta_1 \sim \zeta^{-\delta}$ as $\zeta \to \pm\infty$, where $\delta \geqslant \varkappa_1 > 0$. For this purpose, we have to eliminate the terms $\eta_{0,\tau}$ and $x\eta_{0,\chi}$ from the right side of the equation for η_1 following from Eq. (14). As a result, we obtain the solution $a(\chi,\tau) \equiv a$, $b(\chi,\tau) \equiv b$. This solution leaves open the question on stability of the localized linear wave (17)–(19). To find whether this solution is stable, we have to consider the next terms of series (12) and the dependence of the solution on slower variables.

The localized contour of the function $f(\zeta)$ is formed by a linear combination of the even ($a \neq 0$ and $b = 0$) and odd ($a = 0$ and $b \neq 0$) components. Figure 2 shows the dependence $f(\zeta)$ for the case of a symmetric perturbation ($a = -1$, $b = 0$). For $m > 1$, the solitary wave has a unimodal shape (Fig. 2a), and we obtain a Gaussian function $f(\zeta) = a\exp(-\zeta^2/(2\varkappa_2))$ for $m \to -\infty$. For $m < -1$, the solitary wave near the point $m = -1$ has the form of a group soliton (Fig. 2b). As the parameter m is increased from $m = -1$, the oscillations disappear gradually (Fig. 2c), and we again have a Gaussian function for $m \to -\infty$.

For $m = 1$ (rheology of a linear-viscous fluid), Eq. (18) loses the term $\varkappa_1 f$. The solution of this equation is a kink (localized solution) (Fig. 2d) described by the probability integral $f(\zeta) = b\sqrt{\pi\varkappa_2/2}\,\mathrm{erf}\,(\zeta/\sqrt{2\varkappa_2}\,)$ and does not satisfy the locality boundary conditions (7). If we use the substitution $\eta_0 = f(x - x_*\,\mathrm{e}^t) - f(x + x_*\,\mathrm{e}^t)$ instead of Eq. (17), then we obtain a solution in the form of a distributed, uniformly stretched neck of fixed depth whose edges have an invariable contour and are motionless with respect to the Lagrangian coordinates $\pm x_*$.

The stationary solution of Eq. (18) localized with respect to the self-similar variable can be considered as a result of the balance of the leading terms $\zeta f_{,\zeta}$ and $\varkappa_2 f_{,\zeta\zeta}$. The first term originates from the nonlinear term of the kinematic boundary condition (6), and the second term results from retaining terms of higher powers than the first one in the transverse coordinate in the expressions approximating the velocity components (10) (Porubov [8] also concluded that the hypothesis of plane sections provides no possibility for the existence of localized solutions). The existence of the localized solution is determined by some interval of m, which is the measure of physical nonlinearity

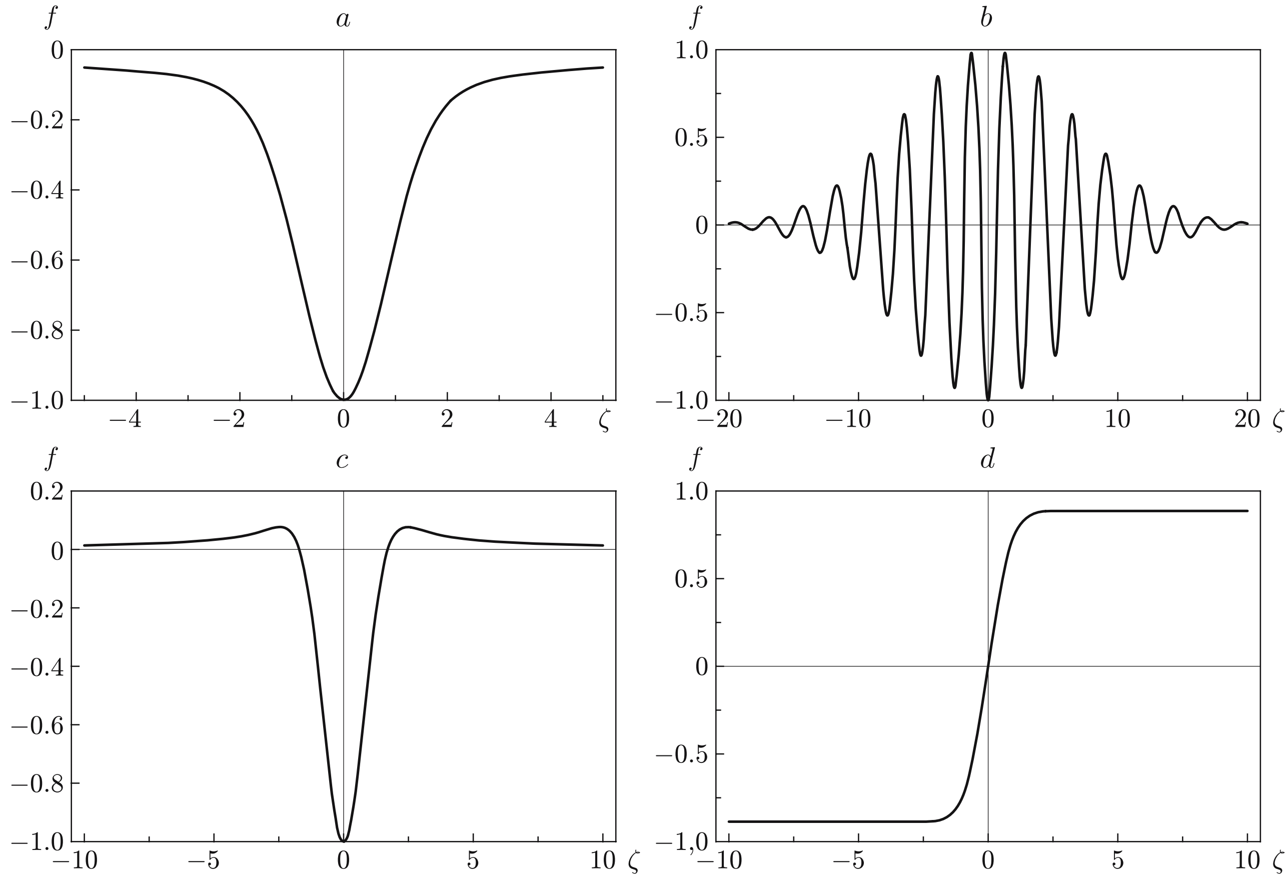

Fig. 2. Shapes of the solitary wave (19): (a–c) $a = -1$, $b = 0$, and $m = 10$ (a), 1.015 (b), and -10 (c); (d) $a = 0$, $b = 1$, and $m = 1$.

of the model. These terms have the lowest order $p = 2$ of the expression from which the expansion of the solution into a power series begins: $f = \exp(-\zeta^p) \sum_{n=0}^{\infty} f_n \zeta^n$.

It follows from Eqs. (10) and (16) that the boundary perturbation is proportional to the perturbation of the elongation rate perturbation of the opposite sign, which is averaged over the band cross section. In particular, the elongation rate increases in the neck region $\eta_0 < 0$.

Solution (19) allows us to estimate the neck shape at $m > 1$. Near the point $\zeta = 0$, we have an expansion into a power series beginning from the terms $f(\zeta) = -1 + \varkappa_1 \zeta^2/(2\varkappa_2) + O(\zeta^4)$. The neck width λ at its half-height is found from the equation $-1 + \varkappa_1 \lambda^2/(8\varkappa_2) \approx -1/2$, whence it follows that $\lambda = 2\sqrt{\varkappa_2/\varkappa_1}$. For values of m slightly greater than unity, we have the asymptotic solution

$$\lambda \sim 2/\sqrt{m-1} + O(\sqrt{m-1}\,) \qquad (20)$$

corresponding to extremely flat necks. For the necks shown in Fig. 1, we have $\lambda \sim 100$ and, according to Eq. (20), $m - 1 \sim 10^{-4}$. In stationary creep experiments (including superplasticity regimes), the most commonly observed values are $0 < m < 1$. Therefore, to identify m on the basis of the current profile of the side surface of the stretched specimen, we have to find other weakly perturbed self-similar contours described by Eq. (15). A systematic search for partial solutions of the differential equation can be performed by studying the structure of its symmetries.

Other Solutions. The general results on the group classification of linear parabolic equations in partial derivatives with two independent variables were obtained by Ovsyannikov and described in [11]. Equation (15) is related to the heat-conduction equation

$$\bar{\eta}_{,\bar{t}} - \bar{\eta}_{,\bar{x}\bar{x}} = 0, \qquad (21)$$

to which it is reduced by the equivalence transformations

$$\bar{\eta} = \mathrm{e}^{\varkappa_1 t}\,\eta_0, \qquad \bar{x} = \varkappa_2^{-1/2}\,\mathrm{e}^{-t}\,x, \qquad \bar{t} = \mathrm{e}^{-2t}/2, \tag{22}$$

preserving the algebraic structure of point symmetries of Eq. (15) [11, 12]. The optimal system of finite-dimensional subalgebras of the infinite-dimensional Lie algebra admitted by Eq. (21) was constructed in [13]. Each of these subalgebras corresponds to a partial solution of Eq. (21), which is related by transformations (22) with a certain (not necessarily localized) evolving self-similar contour of the free boundary. The structure of the Lie algebra in Eq. (21) is independent of the arbitrary parameter of this equation, which is the parameter of sensitivity to the strain rate m. In other words, the problem of the group classification of Eq. (21) with respect to the arbitrary parameter m has a trivial solution. Furthermore, classification and identification of nonlinear-viscous relations is impossible without finding all localized limited solutions in an infinite set of solutions invariant in the corresponding subalgebras of the optimal system and determining their existence conditions depending on the parameter m. Equation (15) is encountered in quantum mechanics; some localized solutions of Eqs. (15) and (21) are given in [12].

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